The quantum superalgebra U_q(osp(1/2n)): deformed para-Bose operators and root of unity representations

The quantum superalgebra $U_q[osp(1/2n)]$: deformed para-Bose operators and root of unity representations

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Abstract. We recall the relation between the Lie superalgebra $osp(1/2n)$ and para-Bose operators. The quantum superalgebra $U_q[osp(1/2n)]$, defined as usual in terms of its Chevalley generators, is shown to be isomorphic to an associative algebra generated by so-called pre-oscillator operators satisfying a number of relations. From these relations, and the analogue with the non-deformed case, one can interpret these pre-oscillator operators as deformed para-Bose operators. Some consequences for $U_q[osp(1/2n)]$ (Cartan–Weyl basis, Poincaré–Birkhoff–Witt basis) and its Hopf subalgebra $U_q[gl(n)]$ are pointed out. Finally, using a realization in terms of 'q-commuting' q-bosons, we construct an irreducible finite-dimensional unitary Fock representation of $U_q[osp(1/2n)]$ and its decomposition in terms of $U_q[gl(n)]$ representations when q is a root of unity.

1. Introduction

It has been established in a number of papers [1–8] that the proper mathematical framework of a system of n para-Bose operators is the theory of (unitarizable) representations of the Lie superalgebra $osp(1/2n)$. This Lie superalgebra is $B(0/n)$ in Kac's notation, and its finite-dimensional irreducible representations (irreps) are completely classified (even their characters are known). Unfortunately, these representations are not unitarizable (see, e.g., [9]). The infinite-dimensional unitarizable representations have not been classified, and so far only certain special cases (corresponding to parastatistics of order p) have been considered.

Following recent interest in q-deformations, a number of papers have dealt with deformed parastatistics and in particular with q-deformed para-Bose operators from different points of view [10–30]. The definition of q-deformed para-Bose operators is highly dependent upon the framework one is working in, and most often one is inspired by considering q-analogues of ordinary Bose operators. Such approaches usually lead to deformed para-Bose operators that are incompatible with a Hopf algebra structure.

From our point of view, the natural *ansatz* is the equivalence between $osp(1/2n)$ irreps and para-Bose operator representations. The q-deformed superalgebra $U_q[osp(1/2n)]$, a Hopf superalgebra, is by now a classical concept [31–36]. Inside $U_q[osp(1/2n)]$ one can in a natural way identify a set of elements with q-deformed para-Bose operators. This

leads to two basic results: an alternative definition of $U_q[osp(1/2n)]$ in terms of non-Chevalley generators (the deformed para-Bose operators) satisfying a number of relations; and the q-analogue of all triple relations defining para-Bose statistics. These relations, in their turn, imply the existence of a Poincaré–Birkhoff–Witt theorem for $U_q[osp(1/2n)]$ and thus a basis in terms of normally ordered monomials of Cartan–Weyl generators, the expressions of which become extremely simple in terms of the deformed para-Bose operators. Apart from these consequences, we also state a number of results for the Hopf subalgebra $U_q[gl(n)] \subset U_q[osp(1/2n)]$ and its realization in terms of the deformed para-Bose operators.

Just as the defining relations for ordinary para-Bose operators are automatically satisfied by canonical Bose operators (leading to the oscillator representation of $osp(1/2n)$), we construct in the present paper a set of deformed Bose operators that satisfy, as a particular example, the deformed para-Bose operator relations. These deformed Bose operators are (up to some factor) equal to the usual q-bosons, but different modes 'q-commute' instead of being commutative. The Fock space for these operators is constructed, and is shown to be unitarizable only if q is a primitive root of unity, in which case the representation becomes finite-dimensional. These finite-dimensional representations are given explicitly, and their decomposition into irreducible representation of $U_q[gl(n)]$ is considered.

2. The para-Bose algebra pB_n and its relation to the Lie superalgebra $osp(1/2n)$

Let $A_i^\pm$ ($i = 1, \ldots, n$) be a system of n para-Bose operators. The defining relation for para-Bose operators (parabosons), introduced in quantum field theory by Green [37] as a possible generalization of integer spin field statistics (see [2] for a general introduction to parastatistics), is given by

$$[\{A_i^\xi, A_j^\eta\}, A_k^\epsilon] = (\epsilon - \eta)\delta_{jk}A_i^\xi + (\epsilon - \xi)\delta_{ik}A_j^\eta \qquad (\xi, \eta, \epsilon = \pm 1). \quad (2.1)$$

The relations (2.1) generalize the canonical commutation relations of ordinary Bose operators (bosons) $a_i^\pm$

$$[a_i^-, a_j^+] = \delta_{ij} \qquad [a_i^-, a_j^-] = [a_i^+, a_j^+] = 0 \quad (2.2)$$

and it is trivial to verify that the $a_i^\pm$ do indeed satisfy (2.1).

The para-Bose algebra pB_n is defined as the associative algebra with unity over $\mathbb{C}$ with generators $A_i^\pm$, subject to the relations (2.1). In fact, pB_n is turned into a superalgebra (associative $\mathbb{Z}_2$-graded algebra) by the requirement $\deg(A_i^\pm) = \bar{1}$, $\forall i \in \{1, \ldots, n\}$, where $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. By defining the supercommutator between any two homogeneous elements a and b of pB_n by

$$[\![a, b]\!] = ab - (-1)^{\deg(a)\deg(b)}ba \quad (2.3)$$

and extending it by bilinearity to the whole algebra, pB_n is turned into a Lie superalgebra. Thus n pairs of para-Bose operators generate a Lie superalgebra [38], and we shall recall that in the present case this Lie superalgebra can be identified with $osp(1/2n)$ [1].

For this purpose, define the Lie superalgebra $osp(1/2n)$ (or $B(0/n)$ in Kac's notation [39]) as the set of $(2n + 1) \times (2n + 1)$ complex matrices of the form

$$\begin{pmatrix} 0 & x & y \\ y^{\mathrm{T}} & d & e \\ -x^{\mathrm{T}} & f & -d^{\mathrm{T}} \end{pmatrix} \quad (2.4)$$

where 'T' stands for transposition, d, e and f are $n \times n$ matrices with $e^T = e$ and $f^T = f$, and x, y are $1 \times n$ matrices. The even subalgebra $osp(1/2n)_{\bar{0}}$ consists of all matrices with $x = y = 0$ and is isomorphic to the symplectic Lie algebra $sp(2n)$. The odd subspace $osp(1/2n)_{\bar{1}}$ consists of all matrices with $d = e = f = 0$. The supercommutator between homogeneous elements is defined by means of (2.3), and extended by bilinearity. Let E_{kl} denote the $(2n+1) \times (2n+1)$ matrix with 1 at the intersection of row k and column l, and zero elsewhere (where rows and columns are labelled from 0 to $2n$). The Cartan subalgebra H of $osp(1/2n)$ is spanned by the elements $H_i = -E_{ii} + E_{n+i,n+i}$ ($i = 1, \ldots, n$). With a suitable basis ε_i ($i = 1, \ldots, n$) of the dual space H^*, the roots of $osp(1/2n)$ consist of $2n$ odd roots $\pm\varepsilon_i$ and $2n^2$ even roots $\pm\varepsilon_i \pm \varepsilon_j$ [39]. The Lie superalgebra $osp(1/2n)$ has the usual root space decomposition, with all root spaces one-dimensional. The root vectors corresponding to the odd roots can be written as follows:

$$\varepsilon_i : \qquad A_i^- = \sqrt{2}(E_{0,i} - E_{i+n,0}) \qquad i = 1, \ldots, n, \qquad (2.5)$$

$$-\varepsilon_i : \qquad A_i^+ = \sqrt{2}(E_{0,i+n} + E_{i,0}) \qquad i = 1, \ldots, n. \qquad (2.6)$$

Therefore the anticommutator $\{A_i^{-\xi}, A_j^{-\eta}\}$ is a root vector with root $\xi\varepsilon_i + \eta\varepsilon_j$ ($\xi, \eta = \pm$), and one finds that $\{A_i^-, A_i^+\} = -2H_i$ ($i, j = 1, \ldots, n$). This implies that the odd root vectors $A_i^{\pm}$ generate the whole Lie superalgebra $osp(1/2n)$. Moreover, one can verify that the elements (2.5)–(2.6) satisfy indeed (2.1). This gives rise to the following proposition.

Proposition 1. The para-Bose algebra pB_n is isomorphic to the universal enveloping algebra $U[osp(1/2n)]$ of $osp(1/2n)$. The finite dimensional subspace

$$\text{span}\{\{A_i^\xi, A_j^\eta\}, A_k^\epsilon | i, j, k = 1, \ldots, n; \; \xi, \eta, \epsilon = \pm\} \qquad (2.7)$$

endowed with the supercommutator (2.3), is a Lie superalgebra isomorphic to $osp(1/2n)$.

Note that the even elements of $osp(1/2n)$ are spanned by the $\{A_i^\xi, A_j^\eta\}$, thus $sp(2n) = \text{span}\{\{A_i^\xi, A_j^\eta\}|i, j = 1, \ldots, n; \; \xi, \eta = \pm\}$. In particular, from (2.1) one can derive a compact expression for the commutation relations of the $sp(2n)$ basis elements:

$$[\{A_i^\xi, A_j^\eta\}, \{A_k^\epsilon, A_l^\varphi\}] = (\epsilon - \eta)\delta_{jk}\{A_i^\xi, A_l^\varphi\} + (\epsilon - \xi)\delta_{ik}\{A_j^\eta, A_l^\varphi\}$$

$$+ (\varphi - \eta)\delta_{jl}\{A_i^\xi, A_k^\epsilon\} + (\varphi - \xi)\delta_{il}\{A_j^\eta, A_k^\epsilon\}. \qquad (2.8)$$

As a consequence of proposition 1, determining all representations of the para-Bose operators is completely equivalent with finding all representations of the Lie superalgebra $osp(1/2n)$. For finite-dimensional irreducible representations of $osp(1/2n)$, there exists a character formula [39], but explicit formulae for the matrix elements are not available in the literature. In para-Bose statistics, one is rather interested in infinite dimensional unitarizable representations. The Fock space corresponding to the ordinary Bose operators (2.2) provides one such example; representations corresponding to a fixed order of parastatistics [37, 40] provide in principle others; but apart from that no general theory of unitarizable representations of $osp(1/2n)$ exists.

Besides the definition of $U[osp(1/2n)]$ in terms of generators $A_i^{\pm}$ subject to the relations (2.1), there is an alternative definition in terms of Chevalley generators. This is perhaps the definition that most readers are more familiar with. Let (α_{ij}) be a Cartan matrix chosen as an $n \times n$ symmetric matrix with

$$\alpha_{nn} = 1 \qquad \alpha_{ii} = 2 \qquad \alpha_{i,i+1} = \alpha_{i+1,i} = -1$$

$$i = 1, \ldots, n-1 \qquad \text{all other } \alpha_{ij} = 0. \qquad (2.9)$$

Again $U[osp(1/2n)]$ is defined as an associative superalgebra in terms of a number of generators subject to relations. The generators are the elements h_i, e_i, f_i $(i = 1, \ldots, n)$; the relations are the Cartan–Kac relations

$$[h_i, h_j] = 0$$
$$[h_i, e_j] = \alpha_{ij} e_j \qquad [h_i, f_j] = -\alpha_{ij} f_j$$
$$[e_i, f_j] = \delta_{ij} h_i \qquad \text{except for } i = j = n \qquad (2.10)$$
$$\{e_n, f_n\} = h_n$$

the e-Serre relations

$$[e_i, e_j] = 0 \qquad \text{for } |i - j| > 1$$
$$e_i^2 e_{i+1} - 2e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0 \qquad i = 1, \ldots, n-1$$
$$e_i^2 e_{i-1} - 2e_i e_{i-1} e_i + e_{i-1} e_i^2 = 0 \qquad i = 2, \ldots, n-1 \qquad (2.11)$$
$$e_n^3 e_{n-1} - (e_n^2 e_{n-1} e_n + e_n e_{n-1} e_n^2) + e_{n-1} e_n^3 = 0$$

and the f-Serre relations obtained from (2.11) by replacing everywhere e_i by f_i. The grading on the superalgebra is induced from the grading on its generators: $\deg(e_n) = \deg(f_n) = \bar{1}$ and $\deg(e_i) = \deg(f_i) = \bar{0}$ for $i = 1, \ldots, n-1$. Although the present definition of $U[osp(1/2n)]$ looks somewhat more complicated than the one given in terms of the para-Bose operators $A_i^\pm$, it cannot be avoided when turning to a Hopf superalgebra deformation $U_q[osp(1/2n)]$. Such deformations are well defined only in a Chevalley basis. In fact, relations like the ones given below, expressing the para-Bose operators in terms of the Chevalley generators, will be reconsidered in the following section to yield a definition of deformed para-Bose operators in terms of the Chevalley generators of $U_q[osp(1/2n)]$. Here, the relations read

$$A_i^- = -\sqrt{2}[e_i, [e_{i+1}, [e_{i+2}, [\ldots, [e_{n-2}, [e_{n-1}, e_n]\ldots] \qquad i = 1, \ldots, n-1$$

$$A_i^+ = \sqrt{2}[\ldots[f_n, f_{n-1}]_q, f_{n-2}], \ldots], f_{i+2}], f_{i+1}], f_i] \qquad i = 1, \ldots, n-1 \qquad (2.12)$$

$$A_n^- = -\sqrt{2}e_n \qquad A_n^+ = \sqrt{2}f_n.$$

Clearly, the expressions of the Chevalley generators in terms of para-Bose operators also exist:

$$e_i = \tfrac{1}{2}\{A_i^-, A_{i+1}^+\} \qquad f_i = \tfrac{1}{2}\{A_i^+, A_{i+1}^-\} \qquad i = 1, \ldots, n-1 \qquad (2.13)$$

$$h_i = \tfrac{1}{2}\{A_{i+1}^-, A_{i+1}^+\} - \{A_i^-, A_i^+\} \qquad i = 1, \ldots, n-1 \qquad (2.14)$$

$$e_n = -\frac{1}{\sqrt{2}}A_n^- \qquad f_n = \frac{1}{\sqrt{2}}A_n^+ \qquad h_n = -\tfrac{1}{2}\{A_n^-, A_n^+\}. \qquad (2.15)$$

An important statement can be deduced from these relations. From equations (2.10)–(2.11) it follows that the enveloping algebra of $gl(n)$ is generated by h_i $(i = 1, \ldots, n)$ and e_i, f_i $(i = 1, \ldots, n-1)$. Then (2.13)–(2.14) and (2.8) show that $U[gl(n)]$, a subalgebra of $U[osp(1/2n)]$, is generated by the elements $\{A_i^-, A_j^+\}$ $(i, j = 1, \ldots, n)$. In other words

$$gl(n) = \text{span}\{\{A_i^-, A_j^+\}|i, j = 1, \ldots, n\} \qquad (2.16)$$

endowed with the usual commutator product. Replacing in (2.16) the para-Bose operators with the ordinary Bose operators (2.2) yields the familiar Schwinger realization of $gl(n)$; but observe that in the superalgebra grading the para-Bose and Bose operators are odd ('fermionic') operators rather than even ('bosonic') operators.

3. The quantum superalgebra $U_q[osp(1/2n)]$ and q-deformed para-Bose operators

In this section we consider the well known quantum superalgebra $U_q[osp(1/2n)]$. A general procedure to construct a so-called Cartan–Weyl basis of $U_q[osp(1/2n)]$ (the analogue of a Cartan–Weyl basis for the Lie superalgebra $osp(1/2n)$) was given in [35]. We follow this procedure, and identify—as in the non-deformed case of section 1—the Cartan–Weyl basis elements corresponding to the odd roots as 'deformed para-Bose operators.' The remaining questions that are treated in this section are: the alternative definition of $U_q[osp(1/2n)]$ in terms of deformed para-Bose operators; the triple relations between deformed para-Bose operators (the analogue of (2.1)); a Poincaré–Birkhoff–Witt (PBW) theorem for $U_q[osp(1/2n)]$; the Cartan–Weyl basis of the subalgebra $U_q[gl(n)]$ and its consequences.

First, we introduce $U_q \equiv U_q[osp(1/2n)]$ by means of its classical definition in terms of the Cartan matrix (2.9). U_q is the associative superalgebra with unity over $\mathbb{C}$, generated by the elements $k_i^{\pm} = q^{\pm h_i}$, e_i, f_i ($i = 1, \ldots, n$), subject to the following relations: the Cartan–Kac relations

$$
\begin{aligned}
&k_i k_i^{-1} = k_i^{-1} k_i = 1 \qquad k_i k_j = k_j k_i \\
&k_i e_j = q^{\alpha_{ij}} e_j k_i \qquad k_i f_j = q^{-\alpha_{ij}} f_j k_i \\
&[e_i, f_j] = \delta_{ij}(k_i - k_i^{-1})/(q - q^{-1}) \qquad \text{except for } i = j = n \\
&\{e_n, f_n\} = (k_n - k_n^{-1})/(q - q^{-1})
\end{aligned}
\tag{3.1}
$$

the e-Serre relations

$$
\begin{aligned}
&[e_i, e_j] = 0 \qquad \text{for } |i - j| > 1 \\
&e_i^2 e_{i+1} - (q + q^{-1}) e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0 \qquad i = 1, \ldots, n-1 \\
&e_i^2 e_{i-1} - (q + q^{-1}) e_i e_{i-1} e_i + e_{i-1} e_i^2 = 0 \qquad i = 2, \ldots, n-1 \\
&e_n^3 e_{n-1} + (1 - q - q^{-1})(e_n^2 e_{n-1} e_n + e_n e_{n-1} e_n^2) + e_{n-1} e_n^3 = 0
\end{aligned}
\tag{3.2}
$$

and the f-Serre relations obtained from above by replacing e_i by f_i everywhere. The grading on U_q is induced from the requirement that the generators e_n, f_n are odd and all other generators are even. It is known that U_q can be endowed with a comultiplication Δ, a counit ε and an antipode S, turning it into a Hopf superalgebra; here we shall not be concerned with this additional structure.

Following the procedure outlined in [35] one determines the Cartan–Weyl elements corresponding to the odd roots, and thus we define the deformed para-Bose operators as follows (see also [41]):

$$
A_i^- = -\sqrt{2}[e_i, [e_{i+1}, [e_{i+2}, [\ldots, [e_{n-2}, [e_{n-1}, e_n]_{q^{-1}}]_{q^{-1}} \ldots]_{q^{-1}}
$$

$$
A_i^+ = \sqrt{2}[\ldots [f_n, f_{n-1}]_q, f_{n-2}]_q, \ldots]_q, f_{i+2}]_q, f_{i+1}]_q, f_i]_q \qquad i = 1, \ldots, n-1 \tag{3.3}
$$

$$
A_n^- = -\sqrt{2} e_n \qquad A_n^+ = \sqrt{2} f_n
$$

where $[u, v]_q = uv - q vu$. Besides these, we also introduce n even 'Cartan' elements

$$
L_i = k_i k_{i+1} \ldots k_n = q^{H_i} \quad \text{where } H_i = h_i + h_{i+1} + \ldots + h_n \qquad i = 1, \ldots, n. \tag{3.4}
$$

We shall call the set of operators $A_i^{\pm}$, $L_i^{\pm}$ pre-oscillator operators, for reasons that will be obvious in the following section. Using their definition (3.3)–(3.4) and the defining relations

(3.1)–(3.2), one can also determine the Chevalley generators in terms of the pre-oscillator operators $(i \neq n)$

$$e_i = -\frac{q}{2}\{A_i^-, A_{i+1}^+\}L_{i+1}^{-1} \qquad f_i = -\frac{1}{2q}L_{i+1}\{A_i^+, A_{i+1}^-\}$$

$$e_n = -(2)^{-1/2}A_n^- \qquad f_n = (2)^{-1/2}A_n^+.$$

(3.5)

Now, one can determine the relations between the pre-oscillator operators. A set of such relations were already obtained in [41], but with the purpose of giving an alternative definition of $U_q[osp(1/2n)]$ in terms of deformed para-Bose or pre-oscillator operators it would be interesting to find the direct analogue of (3.1)–(3.2), and thus present a minimal set of relations. This is given in the following proposition.

Proposition 2. The relations of $U_q[osp(1/2n)]$ in terms of its Chevalley generators hold if and only if the pre-oscillator operators satisfy $(i, j = 1, \ldots, n, \xi = \pm)$

$$L_i L_i^{-1} = L_i^{-1} L_i = 1 \qquad L_i L_j = L_j L_i$$

$$L_i A_j^{\pm} = q^{\mp \delta_{ij}} A_j^{\pm} L_i$$

$$\{A_i^-, A_i^+\} = -2(L_i - L_i^{-1})/(q - q^{-1})$$

$$[\{A_i^{-\xi}, A_{i\pm 1}^{\xi}\}, A_j^{-\xi}]_{q^{\pm \delta_{ij}}} = -2\xi \delta_{j, i\pm 1} L_j^{\pm \xi} A_i^{-\xi}$$

$$[\{A_{n-1}^{\xi}, A_n^{\xi}\}, A_n^{\xi}]]_q = 0.$$

(3.6)

We shall refer to (3.6) as the pre-oscillator realization of U_q. Although the definition of $U_q[osp(1/2n)]$ in terms of deformed para-Bose operators is more complicated than in the non-deformed case, where only one relation (2.1) was necessary, it should be observed that the present definition by means of (3.6) is certainly not more involved than the list of classical relations, i.e. the Cartan–Kac, the e-Serre and the f-Serre relations. The remaining advantage of the definition by means of Chevalley generators is the simplicity of the other Hopf superalgebra functions Δ, ε and S which become very complicated expressions on $A_i^{\pm}$ and $L_i^{\pm 1}$, although they are certainly well defined.

Next, we are concerned with deriving the analogue of (2.1), i.e. all triple relations between deformed para-Bose operators. These relations are derived using (3.6), and they are far more complicated than the classical relation (2.1). In fact, it would be quite impossible to cast all of them in a single expression. Nevertheless, we think they are of importance since they constitute the direct Hopf algebra q-deformation of para-Bose statistics. In the following list, we use the abbreviation:

$$\tau_{i_1, i_2, \ldots, i_k} = \begin{cases} -1 & \text{if } i_1 > i_2 > \cdots > i_k \\ 1 & \text{if } i_1 < i_2 < \cdots < i_k \\ 0 & \text{otherwise.} \end{cases}$$

(3.7)

Then, this list reads:

$$[\{A_i^{-\xi}, A_j^{\xi}\}, A_k^{\xi}]_{q^{\tau_{ji}\delta_{jk}}} = 2\xi \delta_{ik} A_j^{\xi} L_k^{\xi \tau_{ij}} - (q - q^{-1})\tau_{ikj}\{A_i^{-\xi}, A_k^{\xi}\}A_j^{\xi} \qquad (i \neq j)$$

(3.8)

$$[\{A_i^{\xi}, A_j^{\xi}\}, A_k^{-\xi}] = (q - q^{-1})\left(\tau_{kij}\{A_i^{\xi}, A_k^{-\xi}\}A_j^{\xi} + \tau_{kji}\{A_j^{\xi}, A_k^{-\xi}\}A_i^{\xi}\right)$$

$$-2\xi \delta_{ik} A_j^{\xi} L_i^{\xi \tau_{ij}} - 2\xi \delta_{jk} A_i^{\xi} L_j^{\xi \tau_{ji}} \qquad (i \neq j)$$

(3.9)

$$[\{A_i^\xi, A_i^\eta\}, A_k^{-\eta}] = 2\delta_{\xi\eta}(q^{\tau_{ki}} - 1)\{A_i^\xi, A_k^{-\xi}\}A_i^\xi$$

$$-2\xi\eta(1 + \delta_{\xi\eta})\delta_{ik}A_i^\xi \left((q^\xi - 1)L_i^{-1} + (1 - q^{-\xi})L_i\right)/(q - q^{-1}) \tag{3.10}$$

$$[\{A_i^\xi, A_j^\xi\}, A_k^\xi]_{q^{\tau_{ik}+\tau_{jk}}} = 0. \tag{3.11}$$

Apart from giving the q-analogue of para-Bose relations, (3.8)–(3.11) also imply that monomials in $A_i^\pm$ and $\{A_k^\xi, A_i^\eta\}$ can be reordered. A detailed investigation of the quadruple relations, i.e. the relations between $\{A_i^\xi, A_j^\eta\}$ and $\{A_k^\epsilon, A_l^\zeta\}$ (which shall not be given here, since the complete list is too long), yields the following proposition.

Proposition 3. The set of operators

$$L_i^{\pm 1}, \quad A_i^\pm, \quad \{A_i^-, A_j^+\}, \quad \{A_k^\xi, A_l^\xi\} \qquad i \neq j, \; i, j, k, l = 1, \ldots, n, \; \xi = \pm \tag{3.12}$$

give a Cartan–Weyl basis of $U_q[osp(1/2n)]$. The set of all normally ordered monomials [35] in (3.12) constitute a basis in $U_q[osp(1/2n)]$ (PBW theorem).

This shows one of the important advantages of the deformed para-Bose operators: they yield a very simple basis for $U_q[osp(1/2n)]$. At the same time we can restrict the above statements to the subalgebra $U_q[gl(n)]$:

Proposition 4. The operators

$$L_i^{\pm 1} \quad \{A_i^-, A_j^+\} \qquad i \neq j = 1, \ldots, n \tag{3.13}$$

constitute a Cartan–Weyl basis for the Hopf superalgebra $U_q[gl(n)]$; the normally ordered monomials in (3.13) form a basis in $U_q[gl(n)]$.

Let us say a few words about the normal order for the elements (3.12). Usually, one takes an order $<$ such that for positive root vectors (prv) A_i^-, $\{A_i^-, A_j^\pm\}$ $(i < j)$, for negative root vectors (nrv), and for the Cartan generators L_i the inequality $prv < nrv < L_i$ holds. Among the prv the order is taken to be [41]

$$\{A_i^-, A_k^\xi\} < \{A_i^-, A_l^\xi\} \qquad \text{for } k < l \tag{3.14}$$

$$\{A_i^-, A_k^+\} < A_i^- < \{A_i^-, A_l^-\} < \{A_j^-, A_r^+\} < A_j^- < \{A_j^-, A_s^-\}$$

$$\text{for } i < j; \; i < k, l; \; j < r, s. \tag{3.15}$$

To prove proposition 3 one has to show that the unordered product of any two Cartan–Weyl elements (3.12) can be represented as a linear combination of normally ordered products, and that this procedure of ordering is finite when applied to a finite unordered monomial in the Cartan–Weyl elements. As we mentioned previously, this can be deduced from the triple relations (3.8)–(3.11), and from a list of quadruple relations which is too long to be included here. When restricting the elements to (3.13), it turns out that the quadruple relations can also be summarized rather easily, thus yielding a complete proof of proposition 4.

For this purpose, introduce the function

$$\theta_{i_1, i_2, \ldots, i_k} = \begin{cases} 1 & \text{if } i_1 > i_2 > \ldots > i_k \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

A set of Cartan–Weyl elements of $U_q[gl(n)]$ has been considered before [42], and consists of n 'Cartan' elements $L_1, L_2, \ldots, L_n$ and $n(n-1)$ root vectors e_{ij}, $i \neq j = 1, \ldots, n$.

Remember that e_{ij} is positive if $i < j$ and negative if $i > j$. Among the prv, the normal order induced from (3.14)–(3.15) yields

$$e_{ij} < e_{kl} \qquad \text{if } i < k \text{ or } i = k \text{ and } j < l \tag{3.17}$$

for nrv one takes the same rule (3.17), and one chooses $prv < nrv < L_i$. This yields a normal order for the $U_q[gl(n)]$ Cartan–Weyl basis elements. A complete set of relations is given by:

(1) $$L_i^\xi L_j^\eta = L_j^\eta L_i^\xi \qquad L_i e_{jk} = q^{\delta_{ij} - \delta_{ik}} e_{jk} L_i \tag{3.18}$$

(2) For any $e_{ij} > 0$ and $e_{kl} < 0$

$$[e_{ij}, e_{kl}] = \left((q - q^{-1}) \theta_{jkil} e_{kj} e_{il} - \delta_{il} \theta_{jk} e_{kj} + \delta_{jk} \theta_{il} e_{il} \right) L_k L_i^{-1}$$

$$+ L_l L_j^{-1} \left(-(q - q^{-1}) \theta_{kjli} e_{il} e_{kj} - \delta_{il} \theta_{kj} e_{kj} + \delta_{jk} \theta_{li} e_{il} \right)$$

$$+ \delta_{il} \delta_{jk} (L_i L_j^{-1} - L_i^{-1} L_j) / (q - q^{-1}) \tag{3.19}$$

(3) Set $\xi = 1$ if $0 < e_{ij} < e_{kl}$, and $\xi = -1$ if $0 > e_{ij} > e_{kl}$. Then

$$e_{ij} e_{kl} - q^{\xi(\delta_{ik} - \delta_{il} - \delta_{jk} + \delta_{jl})} e_{kl} e_{ij} = \delta_{jk} e_{il} + (q - q^{-1}) \tau_{ijkl} e_{kj} e_{il}. \tag{3.20}$$

To obtain the above relations we have considered equations (3.10)–(3.14) from [42] only for the even generators, first replacing q by $-q$ and then $q^{e_{ii}}$ by L_i. The link with the present operators is now given by

$$e_{ij} = -\tfrac{1}{2} L_j^{-1} \{A_i^-, A_j^+\} \quad \text{for } i < j \qquad \text{and} \qquad e_{ij} = -\tfrac{1}{2} \{A_i^-, A_j^+\} L_i \quad \text{for } i > j. \tag{3.21}$$

It is tedious but straightforward to verify that expressions (3.21) indeed satisfy the relations (3.18)–(3.20).

To conclude this section, observe that for $U_q[osp(1/2n)]$ ($U_q[gl(n)]$), the expressions of the Cartan–Weyl root vectors in terms of the deformed para-Bose operators are the same as the expressions of the Cartan–Weyl basis elements in terms of the non-deformed para-Bose operators for $osp(1/2n)$ ($gl(n)$).

4. Realization of $U_q[osp(1/2n)]$ in terms of deformed Bose oscillators

We have seen that the canonical Bose operators (2.2) satisfy the para-Bose relations (2.1). As a consequence the Fock space built on the Bose operators forms a representation of the para-Bose algebra $pB_n \equiv U[osp(1/2n)]$, usually referred to as the oscillator representation of $osp(1/2n)$. In this section we shall define a new set of operators which can be interpreted as deformed Bose operators. They satisfy the deformed para-Bose relations (3.6) and hence their Fock space forms a representation of $U_q[osp(1/2n)]$, the analogue of the usual oscillator representation. Surprisingly, the unitarity conditions lead to the condition that q should be a root of unity, yielding a finite-dimensional Fock space. This Fock representation is studied in detail, and in particular its decomposition with respect to $U_q[gl(n)]$ is considered, giving rise to certain root of unity representations for this quantum algebra.

Define a set of operators

$$a_i^\pm \qquad \kappa_i = q^{N_i} \qquad i = 1, \dots, n \tag{4.1}$$

satisfying the relations

$$a_i^- a_i^+ - q^{\pm 1} a_i^+ a_i^- = \frac{2}{q^{1/2} + q^{-1/2}} \kappa_i^{\mp 1}$$

$$\kappa_i a_j^\pm = q^{\pm \delta_{ij}} a_j^\pm \kappa_i \tag{4.2}$$

$$a_i^\xi a_j^\eta = q^{\xi\eta} a_j^\eta a_i^\xi \qquad \text{for all } i < j.$$

It is obvious that in the limit $q \to 1$ the operators $a_i^\pm$ reduce to the usual Bose creation and annihilation operators (2.2). For a fixed i the relations coincide (up to a multiple) with the usual so-called q-deformed oscillators [43–46]. Note, however, that the third relation in (4.2) implies that different modes do not commute, but 'q-commute'; such a phenomenon has also been considered before [47–51].

Denote by $W_q(n)$ the associative algebra with unity over $\mathbb{C}$ with generators $a_i^\pm$, $\kappa_i^{\pm 1}$ ($i = 1, \ldots, n$) and relations (4.2). Clearly, $W_q(n)$ is a deformation of the canonical Weyl algebra $W(n)$, generated by n pairs of Bose operators.

Proposition 5. The linear map φ from $U_q[osp(1/2n)]$ into $W_q(n)$, defined on the pre-oscillator generators as

$$\varphi(A_i^\pm) = a_i^\pm \qquad \varphi(L_i) = q^{-1/2}\kappa_i^{-1} \equiv l_i \qquad i = 1, \ldots, n \tag{4.3}$$

and extended on all elements by associativity is a (associative algebra) homomorphism of $U_q[osp(1/2n)]$ onto $W_q(n)$.

The proof follows from the observation that the operators $a_i^\pm$ and $l_i = q^{-1/2}\kappa_i^{-1}$ satisfy equations (3.6). Moreover the generators of $W_q(n)$ are among the images of φ. In particular $\kappa_i = \varphi(q^{-1/2}L_i^{-1})$. From propositions 3 and 5 it follows that the following elements yield an oscillator realization of the Cartan–Weyl generators of $U_q[osp(1/2n)]$

$$l_i^{\pm 1} \quad a_i^\pm \quad \{a_i^-, a_j^+\} \quad \{a_i^\xi, a_j^\xi\} \qquad i \neq j = 1, \ldots, n. \tag{4.4}$$

Deformed Bose creation and annihilation operators have recently been studied [43–45] and in the past [56–59]. The operators (4.2) introduced here coincide with others only in one mode. The main reason for introducing them lies in the underlying connection with the deformed para-Bose operators (3.6). As an important consequence, we shall see that unitary deformed oscillator representations exist only when q is a root of unity.

A representation of $U_q[osp(1/2n)]$ is said to be unitary if the representation space is a Hilbert space and the representatives of $A_i^\pm$ and $L_i^{\pm 1}$ (where $L_i^{\pm 1} = q^{\pm H_i}$ are supposed to be diagonal) satisfy

$$(A_i^+)^\dagger = A_i^- \qquad (H_i)^\dagger = H_i \tag{4.5}$$

where $A^\dagger$ is the Hermitian conjugate to the operator A. In particular, let us now consider the Fock space defined by means of the deformed oscillators (4.2). Requiring (4.5) herein leads to $(a_i^+)^\dagger = a_i^-$, $(N_i)^\dagger = N_i$. But the relations (4.2) remain invariant under this conjugation if and only if $|q| = 1$, i.e. if q is a phase.

Next, we proceed with the construction of the Fock space. As usual, the vacuum vector $|0\rangle$ is defined by means of

$$a_i^- |0\rangle = 0. \tag{4.6}$$

The basis states are of the form

$$|m_1, \ldots, m_n\rangle = \mathcal{N}_{m_1, \ldots, m_n} (a_1^+)^{m_1} \cdots (a_n^+)^{m_n} |0\rangle \tag{4.7}$$

where $\mathcal{N}_{m_1,\ldots,m_n}$ is a normalization constant. Under the unitarity condition, a simple calculation using (4.2) and (4.6) leads to

$$\langle m_1, \ldots, m_n | m_1, \ldots, m_n \rangle = |\mathcal{N}_{m_1,\ldots,m_n}|^2 \alpha(m_1) \cdots \alpha(m_n) \tag{4.8}$$

where

$$\alpha(m) = \frac{(q^{1/2} + q^{-1/2})^m}{2^m [m]_q!} \tag{4.9}$$

and, as usual, $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ and $[x]_q! = [x]_q [x-1]_q \cdots [1]_q$. For all allowed values of the labels m_l, the norm (4.8) should be positive. In particular, this implies that $\alpha(m_i)$ must be positive for all $m_i = 0, 1, \ldots$. Then (4.9) implies that $[m]_q$ should be positive for $m = 0, 1, \ldots$. But since $\bar{q} = e^{i\phi}$ is a pure phase, $[m]_q = \sin(m\phi)/\sin(\phi)$. It follows that the only admissible situation is when q is a primitive root of unity,

$$q = e^{i\pi/k} \tag{4.10}$$

for some positive integer k. In that case, the Fock space is finite dimensional, with

$$m_i \in \{0, 1, \ldots, k-1\} \qquad i = 1, \ldots, n. \tag{4.11}$$

Thus, the total dimension of the Fock space is k^n. When the basis is taken to be orthonormal, i.e.

$$\mathcal{N}_{m_1,\ldots,m_n} = (\alpha(m_1) \cdots \alpha(m_n))^{-1/2} \tag{4.12}$$

the explicit action of the deformed oscillators, using the shorthand notation

$$|m\rangle = |\ldots, m_{i-1}, m_i, m_{i+1}, \ldots\rangle$$
$$|m_i + 1\rangle = |\ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots\rangle \tag{4.13}$$
$$|m_i - 1\rangle = |\ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots\rangle$$

reads

$$\kappa_i |m\rangle = e^{i\pi m_i/k} |m\rangle$$

$$a_i^+ |m\rangle = e^{-i\pi(m_1 + \ldots + m_{i-1})/k} \sqrt{\frac{2\sin(\pi(m_i + 1)/k)\sin(\pi/(2k))}{\sin^2(\pi/k)}} |m_i + 1\rangle \tag{4.14}$$

$$a_i^- |m\rangle = e^{i\pi(m_1 + \ldots + m_{i-1})/k} \sqrt{\frac{2\sin(\pi m_i/k)\sin(\pi/(2k))}{\sin^2(\pi/k)}} |m_i - 1\rangle.$$

It is also interesting to consider the decomposition of the above Fock space representation, irreducible with respect to $U_q[osp(1/2n)]$, according to the quantum subalgebra $U_q[gl(n)]$. From (3.21) we can in fact directly deduce the representatives $\pi(e_{ij})$ of all Cartan–Weyl generators of $U_q[gl(n)]$ in the Fock space:

$$\pi(e_{ij}) = -\cos(\pi/(2k))\kappa_j a_j^+ a_i^- \qquad \text{for } i < j \tag{4.15}$$

$$\pi(e_{ij}) = -\cos(\pi/(2k))a_j^+ a_i^- \kappa_i^{-1} \qquad \text{for } i > j. \tag{4.16}$$

Then, the actual matrix elements follow from (4.14). Note that the subspace spanned by vectors $|m_1, \ldots, m_n\rangle$ with $m_1 + \cdots + m_n = m$ (m constant) in fact forms a submodule for the $U_q[gl(n)]$ action. Since the matrix elements (4.14) of a_i^- (a_i^+) are non-zero for $m_i \neq 0$ ($m_i \neq k$), one deduces that this $U_q[gl(n)]$ module is also irreducible. Thus, the k^n-dimensional irreducible $U_q[osp(1/2n)]$ Fock representation splits into $nk - n + 1$ irreducible $U_q[gl(n)]$ representation (m) characterized by a 'total number' m taking values from 0 up

to $n(k-1)$. The dimension of this irreducible representation (m) is equal to the coefficient of x^m in the expansion of

$$(1 + x + \cdots + x^{k-1})^n = \left(\frac{1 - x^k}{1 - x}\right)^n \qquad (4.17)$$

or, more explicitly,

$$\dim(m) = \sum_{j_0, j_1, \dots, j_{k-1}} \frac{n!}{j_0! j_1! \cdots j_{k-1}!} \qquad (4.18)$$

where the j_i assume all non-negative integer values such that $j_0 + j_1 + \cdots + j_{k-1} = n$ and $j_1 + 2j_2 + \cdots + (k-1)j_{k-1} = m$. Of course, the representations (m) obtained by means of deformed Bose operators are only a small part of the so-called type 1 representations of $U_q[gl(n)]$ (see, e.g., chapter 11 of [60], and references therein).

Acknowledgments

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